Stress Predictions of the Mooney–Rivlin Hyperelastic Model Under Uniaxial Tension and Simple Shear

Siyi Li

Department of Chemical, Paper, and Biomedical Engineering

Miami University

Abstract

Hyperelastic constitutive models are widely used to describe large, nonlinear deformations in elastomeric and soft biological materials. In this study, the mechanical behavior of an incompressible Mooney–Rivlin material is evaluated under two fundamental loading paths: uniaxial tension and simple shear. The strain energy density function is used to derive the First Piola–Kirchhoff (engineering) stress for each deformation mode, and stress–strain curves are generated for three representative parameter sets. The uniaxial results exhibit nonlinear strain-stiffening behavior, while the shear response remains strictly linear due to the dependence $P_{12} = 2(C_1 + C_2)\gamma$. Variations in material parameters produce predictable changes in stiffness across both loading cases. These findings illustrate how the Mooney–Rivlin model captures essential features of soft material mechanics and highlight the relative roles of C_1 and C_2 in governing tensile and shear behavior.

1 Introduction

Hyperelastic material models are widely used to represent the large-deformation mechanical behavior of elastomeric and soft biological tissues. Unlike linear elastic models, hyperelastic constitutive laws describe the material response through a strain energy density function W, from which stresses under arbitrary loading paths can be derived. These models are particularly useful for simulating nonlinear deformation, incompressibility, and strain-stiffening behaviors that appear in rubbers, silicone samples, and soft tissues.

Among hyperelastic formulations, the Mooney–Rivlin model is one of the most commonly used phenomenological models for incompressible materials. It expresses the strain energy in terms of the first and second invariants of the deformation tensor and provides a simple two-parameter description of nonlinear elasticity. Despite its simplicity, the Mooney–Rivlin model can capture key features of soft material responses, including nonlinear stress–stretch behavior in tension and linear shear stiffness.

The goal of this project is to explore the mechanical predictions of the Mooney–Rivlin model under two fundamental deformation modes: uniaxial tension and simple shear. For each loading case, the appropriate deformation gradient is defined, the corresponding First Piola–Kirchhoff (engineering) stress is derived, and stress–strain curves are generated for three representative parameter sets. These results illustrate how the material constants influence stiffness in different loading states and provide insight into the capabilities and limitations of the Mooney–Rivlin formulation.

2 Constitutive Model

The Mooney–Rivlin hyperelastic model describes the strain energy of an incompressible isotropic solid using a two-parameter formulation. The strain energy density function is expressed in terms of the first and second invariants of the right Cauchy–Green deformation tensor $C = F^T F$:

$$W = C_1(I_1 - 3) + C_2(I_2 - 3),$$

where C_1 and C_2 are material constants, and

$$I_1 = tr(\mathbf{C}),$$

$$I_2 = \frac{1}{2}[(tr\mathbf{C})^2 - tr(\mathbf{C}^2)].$$

Because the material is treated as incompressible, the determinant of the deformation gradient satisfies:

$$J = det\mathbf{F} = 1.$$

The First Piola–Kirchhoff (PK1) stress, which also represents engineering stress and is directly comparable to experimentally measured F/A_0, is obtained from the strain energy through

$$\mathbf{P} = \frac{\delta W}{\delta \mathbf{F}}.$$

To evaluate the model under different loading paths, we specify the deformation gradient **F** for each case and compute the corresponding stress component.

2.1 Uniaxial Tension

For uniaxial extension along the x-direction, the principal stretch is

$$\lambda_1 = \lambda,$$

and incompressibility requires

$$\lambda_2 = \lambda_3 = \lambda^{-1/2}.$$

The deformation gradient is therefore

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix}.$$

Substituting into the Mooney–Rivlin strain energy function and differentiating with respect to λ gives the First Piola–Kirchhoff stress:

$$P(\lambda) = 2C_1(\lambda - \lambda^{-2}) + 2C_2(1 - \lambda^{-3}).$$

This stress is plotted against stretch λ to obtain the uniaxial stress–stretch response.

2.2 Simple Shear

For simple shear with shear strain γ, the deformation gradient is

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The first and second invariants become:

$$I_1 = I_2 = 3 + \gamma^2.$$

Substituting into the strain energy and differentiating yields the PK1 shear stress:

$$P_{12}(\gamma) = 2(C_1 + C_2)\gamma$$

which predicts a linear shear stress–strain response for Mooney–Rivlin materials.

3 Results

3.1 Uniaxial Tension

The predicted First Piola–Kirchhoff (engineering) stress–stretch curves for the three Mooney–Rivlin parameter sets are shown in Figure 1. All curves exhibit a nonlinear increase in stress with stretch, consistent with the characteristic strain-stiffening behavior of incompressible hyperelastic materials. As expected, parameter sets with larger values of C_1 or C_2 generate higher stresses for a given stretch. **Set 2**, which has the largest C_1, produces the stiffest response across the entire range of λ. **Set 3**, with a moderately larger C_2, also exhibits increased stiffness relative to Set 1, especially at larger stretches where the nonlinear terms dominate. These trends illustrate how the Mooney–Rivlin parameters independently and jointly influence tensile resistance.

3.2 Simple Shear

Figure 2 shows the engineering shear stress P_{12} as a function of shear strain γ. In contrast to uniaxial tension, all three parameter sets produce strictly linear stress–strain relationships. This result directly follows from the analytical expression $P_{12} = 2(C_1 + C_2)\gamma$, which predicts a constant shear modulus determined solely by the sum of the material parameters. The relative ordering of stiffness between sets is consistent with the values of $C_1 + C_2$:

- **Set 2** has the highest combined parameter value and shows the steepest slope.
- **Set 3** shows moderate stiffness.
- **Set 1** produces the lowest shear stress for a given strain.

These results highlight that, within the Mooney–Rivlin model, shear behavior depends only on the combined parameter $C_1 + C_2$ and lacks the nonlinear stiffening observed in uniaxial tension.

4 Discussion

The results highlight several important features of the Mooney–Rivlin hyperelastic formulation. First, the material demonstrates clear strain-stiffening behavior in uniaxial tension. This arises from the nonlinear dependence of the PK1 stress on the stretch, particularly through terms involving λ^{-2} and λ^2. Materials with higher C_1 or C_2 exhibit larger stresses throughout the loading range, confirming that both parameters contribute to tensile stiffness, with C_2 influencing the response more prominently at larger strains.

In contrast, the simple shear results show that the model predicts a strictly linear shear stress–strain relationship. This follows directly from the analytical expression $P_{12} = 2(C_1 + C_2)\gamma$, which indicates that shear stiffness is governed only by the sum of the two coefficients rather than their individual contributions. As a result, parameter sets with the same value of $C_1 + C_2$ would produce identical shear responses, even if their uniaxial behavior differs.

The Mooney–Rivlin model is widely used to represent soft biological tissues—such as arteries, myocardium, and dermal tissues—because these materials undergo large, incompressible deformations similar to elastomers. While real tissues often require more advanced models to capture fiber reinforcement or anisotropy, the Mooney–Rivlin formulation provides a useful baseline for understanding nonlinear elasticity and for comparing loading paths such as tension and shear.

Together, these findings demonstrate that the Mooney–Rivlin model can capture both nonlinear tensile stiffening and linear shear stiffness in incompressible materials. However, the model's simplicity also presents limitations: it may not fully describe materials with strong nonlinear shear behavior, anisotropy, or rate-dependent effects. Nevertheless, the results from these two loading paths provide clear insight into how the model parameters influence the mechanical response and how the constitutive form dictates differences between deformation modes.

5 Conclusion

This project examined the mechanical predictions of the Mooney–Rivlin hyperelastic model under uniaxial tension and simple shear. By deriving the First Piola–Kirchhoff stresses for each loading path and evaluating multiple parameter sets, the results demonstrated nonlinear strain-stiffening behavior in tension and linear stress–strain behavior in shear. The influence of the material parameters was clearly observed, with increased values of C_1 and C_2 producing stiffer responses. These findings highlight both the capabilities and the limits of the Mooney–Rivlin formulation in modeling incompressible, soft elastic materials.

References

[1] HOLZAPFEL, G. A. (2000). *Nonlinear Solid Mechanics: A Continuum Approach for Engineering. John Wiley & Sons.*

[2] OGDEN, R. W. (1997). *Non-Linear Elastic Deformations. Dover Publications.*

[3] BONET, J., & WOOD, R. D. (2008). *Nonlinear Continuum Mechanics for Finite Element Analysis (2nd ed.). Cambridge University Press.*

[4] MOONEY, M. (1940). *A theory of large elastic deformation. Journal of Applied Physics, 11(9), 582–592.*

[5] RIVLIN, R. S. (1948). *Large elastic deformations of isotropic materials. Philosophical Transactions of the Royal Society A, 241(835), 379–397.*

[6] FUNG, Y. C. (1993). *Biomechanics: Mechanical Properties of Living Tissues (2nd ed.). Springer.*

6 Appendix: Figures

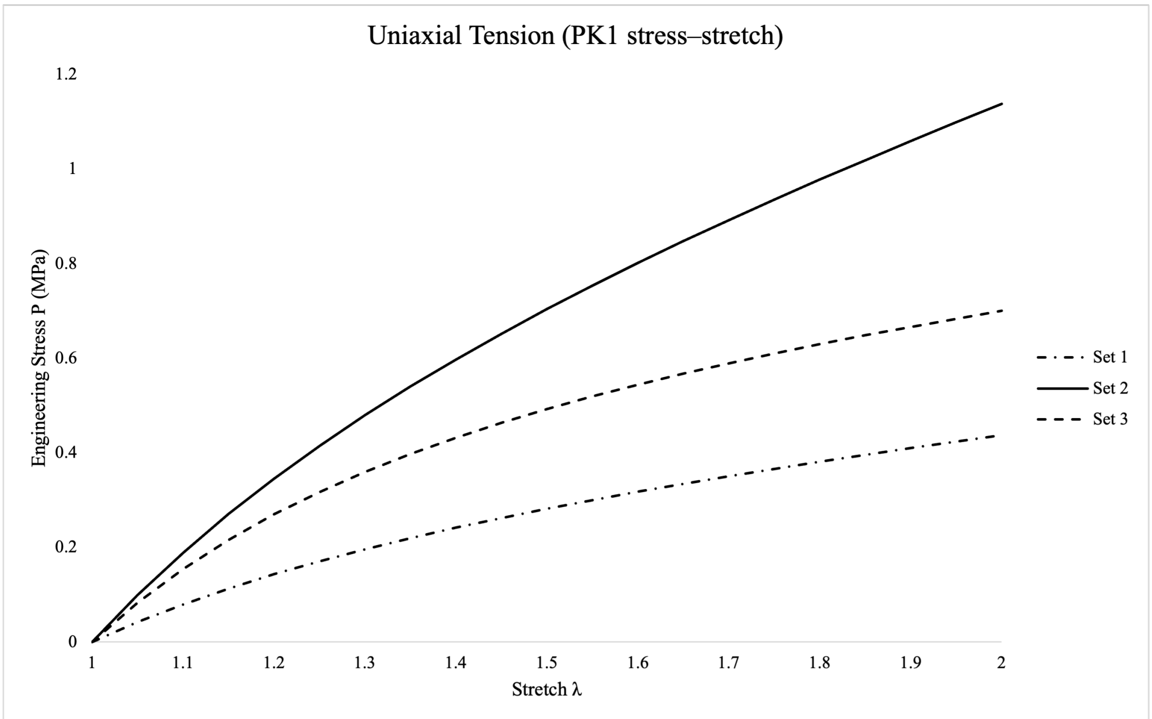

Figure 1: Engineering (PK1) stress–stretch response for three Mooney–Rivlin material parameter sets under uniaxial tension. The curves show the nonlinear increase in tensile stress with stretch, with higher values of C_1 and C_2 producing stiffer responses. Set 2 exhibits the highest stiffness across the loading range due to its larger C_1, while Set 1 is the softest. These results illustrate the strain-stiffening characteristics of the Mooney–Rivlin model.

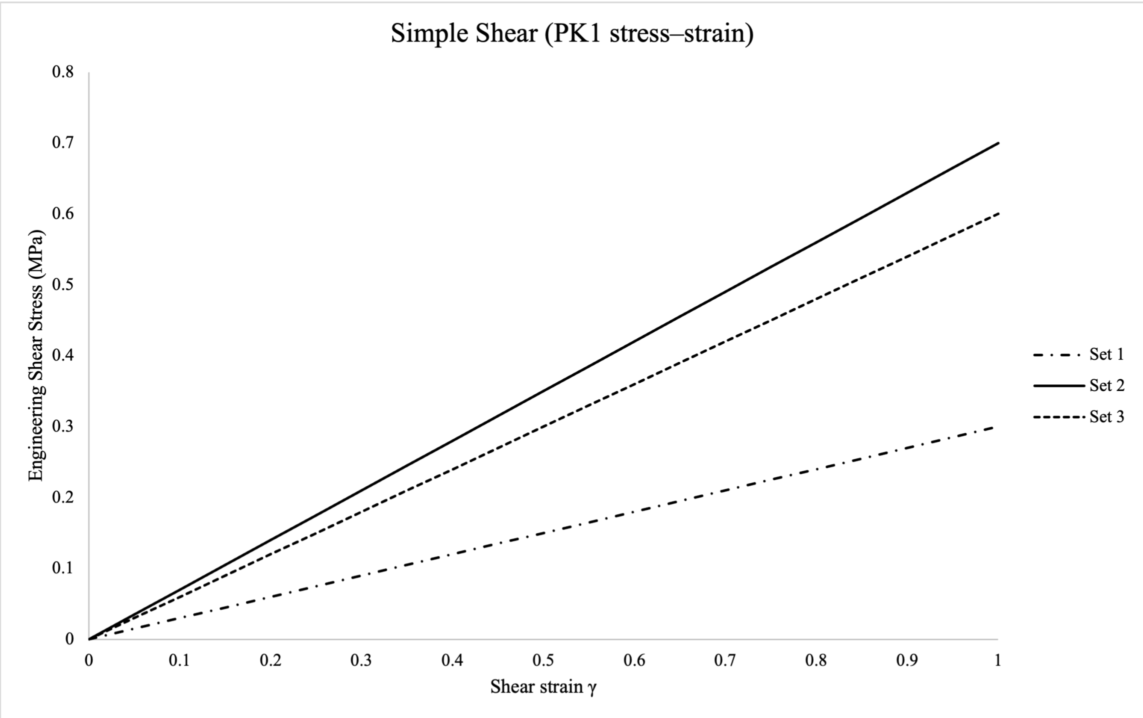

Figure 2: Engineering shear stress P_{12} versus shear strain γ for three Mooney–Rivlin material parameter sets under simple shear. All three curves follow linear trends predicted by $P_{12} = 2(C_1 + C_2)\gamma$ reflecting a constant shear modulus determined by the sum $C_1 + C_2$. Set 2, which has the largest $C_1 + C_2$, displays the steepest slope, whereas Set 1 is the softest.